1. Find the derivative of the following.
   (a) $e^{e^{e^x}}$
   (b) $\log(\log(\log x^2)^2)$
   (c) $\sin(\tan(\cos x))$

2. (a) Let f, g be positive differentiable functions. Find the derivative of
$$h(x) = f(x)^{g(x)}.$$
   (b) Hence find the derivative of x^{x^x}.

3. Use the chain rule to prove the product rule. **Hint:** It is easy to verify that for all a, b we have
$$ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$$

4. Find the equation of the tangent line of $f(x)$ at $x = 1$ given that x satisfies the functional equation
$$f(x^2) = 4f(x) + x^2.$$

5. This is a solution to the bonus problem I gave in class. The goal of this problem is to solve the differential equation
$$y' + 2y = 1 \quad y(0) = 1.$$
   (a) Let's multiply both sides some $f(x)$ to get
$$f(x)y'(x) + 2f(x)y(x) = f(x).$$
   What condition must left hand side satisfy for it to equal $(fy)'$?
   (b) Show that $f(x) = e^{2x}$ is a suitable function.
   (c) Use (a) and (b) to find $y(x)$.

Solution.

1. (a) We solve by repeated chain rule.

$$\left(e^{e^{e^x}}\right)' = e^{e^{e^x}}\left(e^{e^x}\right)' = e^{e^{e^x}} e^{e^x} (e^x)' = e^{e^{e^x}} e^{e^x} e^x$$

(b) First let's simplify.

$$f(x) = \log(\log(\log x^2)^2) = \log(\log(2\log x)^2) = 2\log(\log(2\log x))$$

So now by chain rule

$$\begin{aligned} f'(x) =& 2\frac{1}{\log(2\log x)}(\log(2\log x))' \\ =& 2\frac{1}{\log(2\log x)}\frac{1}{2\log x}(\log x)' \\ =& 2\frac{1}{\log(2\log x)}\frac{1}{2\log x}\frac{1}{x} \\ =& \frac{1}{x\log x\log(2\log x)} \end{aligned}$$

(c) Again by chain rule

$$\begin{aligned} (\sin(\tan(\cos x)))' =& \cos(\tan(\cos x))(\tan(\cos x))' \\ =& \cos(\tan(\cos x))\sec^2(\cos x)(\cos x)' \\ =& -\cos(\tan(\cos x))\sec^2(\cos x)\sin x. \end{aligned}$$

2. (a) Let $y = f^g$, so by taking the log of both sides we get

$$\log y = g\log f.$$

Now let's take derivatives.

$$\begin{aligned} \frac{y'}{y} &= g'\log f + g\frac{f'}{f} \\ y' &= y\left(g'\log f + \frac{gf'}{f}\right) \\ y' &= f^g\left(g'\log f + \frac{gf'}{f}\right) \end{aligned}$$

(b) if $y(x) = x^{x^x} = f(x)^g(x)$, where $f(x) = x$ and $g(x) = x^x$. We have $f' = 1$ and by (a) we have

$$g'(x) = x^x\left(1\cdot\log x + \frac{x\cdot 1}{x}\right) = x^x(\log x + 1)$$

Now again by (a),

$$\begin{aligned}
\left(x^{x^x}\right)' =& f^g\left(g'\log f + \frac{gf'}{f}\right)\\
=& x^{x^x}\left(x^x(\log x + 1)\log x + \frac{x^x\cdot 1}{x}\right)\\
=& x^{x^x}\left(x^x(\log x + 1)\log x + x^{x-1}\right)
\end{aligned}$$

3. By the hint we have

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2].$$

By taking derivatives we have

$$\begin{aligned}
(fg)' =& \frac{1}{4}[2(f+g)(f+g)' - 2(f-g)(f-g)']\\
=& \frac{1}{2}[(f+g)(f'+g') - (f-g)(f'-g')]\\
=& \frac{1}{2}[ff' + fg' + f'g + gg' - ff' + gf' + f'g - gg']\\
=& \frac{1}{2}[2f'g + 2fg']\\
=& f'g + gf'
\end{aligned}$$

4. Recall the equation of the tangent line of $f(x)$ at $x = a$ is

$$y = f(a) + f'(a)(x-a).$$

So let's figure out $f(1), f'(1)$. To find $f(1)$ we plug 1 into the functional equation for f to get

$$f(1) = 4f(1) + 1$$

Solving for $f(1)$ we get $f(1) = -1/3$. To get $f'(1)$ let's take the derivative, we get by chain rule

$$f'(x^2)2x = 4f'(x) + 2x.$$

So by plugging in $x = 1$ we get,

$$2f'(1) = 4f'(1) + 2$$

Which implies $f'(1) = -1$. Thus the equation of the tangent line is

$$y = -\frac{1}{3} - (x-1) = \frac{2}{3} - x$$

5. (a) If we multiply both sides of the equation by f we get

$$fy' + 2fy = f.$$

As suggested by the hint lefts equate the left hand side to $(fy)'$ to get

$$\begin{aligned}(fy)' =& fy + 2fy \\ fy' + yf' =& fy' + 2fy\end{aligned}$$

Which is true if $f' = 2f$.

(b) We know from class that $f = Ae^{2x}$ is the solution to $f' = 2f$. So by multiplying the equation by f we get

$$\begin{gathered}Ae^{2x}y' + A2e^{2x}y = Ae^{2x} \\ e^{2x}y' + 2e^{2x}y = e^{2x}\end{gathered}$$

So we can just assume that $f = e^{2x}$.

(c) By (b) we have

$$e^{2x}y' + 2e^{2x}y = e^{2x}$$

By (a) we chose f so that the above equation would satisfy

$$(e^{2x}y)' = e^{2x}.$$

So $(e^{2x}y)$ equal a function whose derivative is e^{2x}. With a bit of thought one can verify that for all C.

$$\left[\frac{1}{2}e^{2x} + C\right]' = e^{2x}$$

So we have

$$e^{2x}y = \frac{1}{2}e^{2x} + C$$

And thus

$$y(x) = \frac{1}{2} + Ce^{-2x}$$

Finally since $y(0) = 1$ we have

$$1 = \frac{1}{2} + C,$$

So $C = \frac{1}{2}$ and

$$y(x) = \frac{1}{2} + \frac{1}{2}e^{-2x}.$$